QUALIFYING EXAM: PDE I, JANUARY 2021.

- Please write your solutions neatly and legibly. **Please write the response to each question on a separate page and clearly indicate what question you are answering.**
- This exam is open book: you may use textbooks, course notes, old HW and exams. You may NOT consult with others during the exam. Doing so will be in violation of MSU's policy on academic integrity:
`https://ombud.msu.edu/resources-self-help/academic-integrity`.
- You can apply known theorems and facts from class or the book. When doing so, be sure to state the name as best you can (eg, "maximum principle," or "representation formula for solutions to the heat equation.")
- If in doubt, provide more, rather than less, details of proofs or computations.

Problems. There are 5 problems. Each is worth 10 points.

(1) Suppose u is harmonic on $\mathbb{R}^n$ and satisfies, for some constant $C > 0$ and for all $x \in \mathbb{R}^n$,

$$\int_{\{y:|y-x|<1\}} |u(y)|\, dy \leq C.$$

Prove that u is constant.

(2) Let $U \subset \mathbb{R}^n$ be open and bounded, with smooth boundary and let $c \in C^2(U)$. Suppose $u \in C(\bar{U}) \cap C^2(U)$ satisfies,

$$-\Delta u + uc = 0 \text{ in } U.$$

(a) (5 points) Suppose $c(x) \geq 0$ for all $x \in U$. Prove

(*) $$\max_{\bar{U}} u \leq \max_{\partial U} u^+,$$

where u^+ is defined by, $u^+(x) = \max\{u(x), 0\}$.

(b) (5 points) Does the estimate (*) from part (a) necessarily hold without the assumption $c \geq 0$? Give a proof or counterexample.

(3) Let $g \in C(\mathbb{R})$ be such that

$$\lim_{x\to\infty} g(x) = b, \lim_{x\to-\infty} g(x) = c,$$

for some $b, c \in \mathbb{R}$. Let u be a bounded solution to,

$$\begin{cases} u_t - u_{xx} = 0 \text{ in } \mathbb{R} \times (0, \infty), \\ u(x, 0) = g(x) \text{ on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

Use the representation formula for u in terms of the fundamental solution to the heat equation in order to find $\lim_{t\to\infty} u(x, t)$. Justify your answer. (You do not, however, need to provide justification for "legal" exchanges of limit and integral. Also, you may use the fact that $\int_0^\infty e^{-z^2} = \sqrt{\pi}/2$.)

(4) Let $U \subset \mathbb{R}^n$ be bounded and open with smooth boundary. Consider the system

$$\begin{cases} u_{tt} - \Delta u = 0 \text{ on } U \times (0,T), \\ u(x,0) = u_t(x,0) = 0 \text{ for } x \in U, \\ u_t + \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial U \times [0,T], \end{cases}$$

where $\frac{\partial u}{\partial \nu}$ denotes the outward normal derivative of u on ∂U. Prove, using energy methods, that if $u \in C^2(\bar{U} \times [0,T])$ satisfies this system, then $u \equiv 0$ on $U \times [0,T]$.

(5) (a) (7 points) Use the method of characteristics to find a solution u to,

$$\begin{cases} u_t = x u_x - u \text{ on } \mathbb{R} \times (0,\infty), \\ u(x,0) = f(x) \text{ on } \mathbb{R} \times \{0\}, \end{cases}$$

where f is a given smooth function on $\mathbb{R}$.

(b) (3 points) Suppose f has compact support. Determine the limit as $t \to \infty$ of the solution u that you found in the previous part.